ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Differential Equations 204 (2004) 292–322

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Evolution of a semilinear parabolic system for migration and selection in population genetics

Yuan Lou[a,*] and Thomas Nagylaki[b]

[a] *Department of Mathematics, The Ohio State University, 231 W. 18th Avenue, Columbus, OH 43210, USA*

[b] *Department of Ecology and Evolution, The University of Chicago, 1101 East 57th Street, Chicago, IL 60637, USA*

Received August 21, 2003; revised December 31, 2003

Abstract

The semilinear parabolic system that describes the evolution of the gene frequencies in the diffusion approximation for migration and selection at a multiallelic locus is investigated. The population occupies a finite habitat of arbitrary dimensionality and shape (i.e., a bounded, open domain in R^d). The selection coefficients depend on position and may depend on the gene frequencies; the drift and diffusion coefficients may depend on position. Sufficient conditions are given for the global loss of an allele and for its protection from loss. A sufficient condition for the existence of at least one internal equilibrium is also offered, and the profile of any internal equilibrium in the zero-migration limit is obtained.

MSC: 35B40; 35K57; 92D10; 92D25

Keywords: Partial differential equations; Clines; Polymorphism; Geographical variation

1. Introduction

In a recent paper [17], we described the diffusion approximation for migration and selection at a multiallelic locus, outlined its biological background and significance (including a review of the literature), and began its analysis. For two alleles (the scalar case), we extended Henry's [9, pp. 314–319] global analysis from homogeneous, isotropic migration (corresponding to the Laplacian) to arbitrary

*Corresponding author. Fax: 614-292-1479.

E-mail address: lou@math.ohio-state.edu (Y. Lou).

0022-0396/$ - see front matter
doi:10.1016/j.jde.2004.01.009

migration (corresponding to an arbitrary elliptic operator). For multiple alleles (corresponding to a parabolic system), we gave sufficient conditions for the global loss of an allele that is nowhere the fittest.

In Section 2, we greatly generalize the sufficient conditions in [17] for global loss of an intermediate allele and present more explicit sufficient conditions in the case of weak migration. Section 3 is devoted to sufficient conditions for the protection of an allele and for the existence of an internal equilibrium and to the zero-migration limit of any internal equilibrium. In Section 4, we specialize to the natural, important case of frequency-independent selection: we offer general results, a modification of our first main theorem, and several illuminating, significant examples. In Section 5, we briefly discuss strong and intermediate migration. Two propositions stated and proved in the appendix provide insight and may facilitate some analyses.

We posit local random mating, weak viability selection, and weak genotype-independent migration that satisfies the standard assumptions for a diffusion process. We denote position in the finite habitat Ω (a bounded, open domain in R^d) by the vector $x=(x_1,\ldots,x_d)$ and measure time, t, in generations. The population density at x is $\rho(x)$. Let $M_\alpha(x)$ and $V_{\alpha\beta}(x)$ designate the mean displacement in direction x_α and the covariance of the displacements in directions x_α and x_β per generation; these drift and diffusion coefficients form the vector $M(x)$ and the symmetric, positive definite matrix $V(x)$. We consider a single locus with alleles A_i, where $i\in N\equiv\{1,2,\ldots,n\}$. Note that i and j refer to alleles, whereas α and β refer to spatial components.

Let $p_i(x,t)$ signify the (relative) frequency of A_i at position x at time t. For every $x\in\bar{\Omega}$ and $t\geqslant 0$, the vector $p(x,t)$ must satisfy

$$p(x,t)\in\Delta\equiv\left\{p\in R^n:\ p_i\geqslant 0\quad\forall i,\ \sum_{j=1}^{n}p_j=1\right\}. \tag{1.1}$$

To avoid the trivial case of essentially absent alleles, we suppose

$$\int_\Omega p_i(x,0)\,dx>0 \tag{1.2}$$

for every $i\in N$. The (scaled) selection coefficient of the genotype A_iA_j is the function $r_{ij}(x,p)$. Of course, $r_{ij}(x,p)=r_{ji}(x,p)$ for every i, j, x, and p. Let $r_i(x,p)$ and $\bar{r}(x,p)$ represent the selection coefficient of A_i and the mean selection coefficient of the population, respectively:

$$r_i(x,p)=\sum_{j=1}^{n}r_{ij}(x,p)p_j,\quad \bar{r}(x,p)=\sum_{i=1}^{n}r_i(x,p)p_i. \tag{1.3}$$

The contribution of selection is

$$S_i(x,p)\equiv\lambda p_i(r_i-\bar{r}), \tag{1.4}$$

where λ denotes the selection intensity (which we did not factor out in [17]). If time is scaled so that the migration rates are of order one, then λ becomes the ratio of the strength of selection to that of migration. Thus, weak, intermediate, and strong migration relative to selection correspond respectively to large, intermediate, and small values of λ compared with 1.

We define the divergence of an arbitrary symmetric matrix $W(x)$ as the vector with components ($\alpha = 1, 2, \ldots, d$)

$$(\nabla \cdot W)_\alpha = \sum_{\beta=1}^{d} W_{\alpha\beta,x_\beta}, \tag{1.5}$$

where the subscript x_β indicates partial differentiation. We introduce the vector

$$b(x) = \rho^{-1}\nabla \cdot (\rho V) - M \tag{1.6}$$

and the operators L and B defined by

$$Lu = \frac{1}{2}\sum_{\alpha,\beta=1}^{d} V_{\alpha\beta}u_{x_\alpha x_\beta} + b \cdot \nabla u, \tag{1.7a}$$

$$Bu = v \cdot V\nabla u, \tag{1.7b}$$

where v denotes the unit outward normal vector on the boundary $\partial\Omega$.

The gene frequencies $p(x,t)$ satisfy the semilinear parabolic system [18–20]:

$$p_{i,t} = Lp_i + S_i(x,p) \quad \text{in } \Omega \times (0,\infty), \tag{1.8a}$$

$$Bp_i = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.8b}$$

$$p(x,0) \in \Delta \quad \text{in } \bar{\Omega} \tag{1.8c}$$

for every $i \in N$. Here, L describes migration and (1.8b) specifies that no individuals cross the boundary. We are given $\rho(x)$, $M(x)$, $V(x)$, λ, $r_{ij}(x,p)$, and $p(x,0)$; we seek the asymptotic behavior of $p(x,t)$ at $t \to \infty$.

Throughout this paper, we assume that $\rho(x)$, $M_\alpha(x)$, and $V_{\alpha\beta}(x)$ are all Hölder continuous functions; $r_{ij}(x,p)$ are Hölder continuous in x and Lipschitz continuous in p; and $p_i(x,0)$ are continuous functions. We also assume that $\partial\Omega \in C^2$. By the maximum principle [23, Chapter 3] and the standard existence theory of evolution equations, the problem (1.8) has a unique classical solution $p(x,t)$ that exists for all time, $p_i \in C(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$, and $0 < p_i(x,t) < 1$ for every $i \in N$, every $x \in \bar{\Omega}$, and $t > 0$. Therefore, without loss of generality, we posit that $0 < p_i(x,0) < 1$ for every $i \in N$ and every $x \in \bar{\Omega}$. This problem makes sense, i.e., (1.1) holds [17].

We now briefly recapitulate the pertinent results in [17], which can be consulted for discussion and precise statements. Suppose that there is no frequency

dependence, i.e.,

$$r_{ij} = r_{ij}(x) \tag{1.9}$$

for every $i,j \in N$. We assume that

(a1) there exist $i \in N$ and a constant $\gamma_i \in [0,1]$ such that

$$\gamma_i r_{1j}(x) + (1-\gamma_i) r_{nj}(x) \geqslant r_{ij}(x) \tag{1.10}$$

for every $j \in N$ and every $x \in \bar{\Omega}$, and that inequality (1.10) is strict for some $j \in N$ and some $x \in \bar{\Omega}$.

Theorem 3.1 of [17] informs us that *the assumption* (a1) *implies the elimination of* A_i *from the population*: $p_i(x,t) \to 0$ *uniformly in* x *as* $t \to \infty$.

The other multiallelic results in [17] are based on the hypothesis that there is no dominance: there exist smooth selection coefficients $s_i(x)$ such that

$$r_{ij}(x) = s_i(x) + s_j(x) \tag{1.11}$$

for every $i,j \in N$ and every $x \in \bar{\Omega}$. Then we have ($p \in R^n$)

$$S_i(x,p) = \lambda p_i [s_i(x) - \bar{s}(x,p)], \quad \bar{s}(x,p) = \sum_{i=1}^{n} s_i(x) p_i. \tag{1.12}$$

Eq. (1.10) now simplifies to

$$\gamma_i s_1(x) + (1-\gamma_i) s_n(x) \geqslant s_i(x) \tag{1.13}$$

for every $x \in \bar{\Omega}$; here we do not assume that inequality (1.13) is strict for some $x \in \bar{\Omega}$. We posit that

(a2) the selective difference $s_1(x) - s_n(x)$ changes sign;

(a3) there exist constants $\gamma_i \in (0,1)$ such that (1.13) holds for every $i \in \tilde{N} \equiv \{2,3,\ldots,n-1\}$ and every $x \in \bar{\Omega}$.

Then Theorems 3.2 and 3.3 of [17] show that *there is always a globally attracting equilibrium and determine when it is* $p_1 = 1$, $p_n = 1$, *or a diallelic polymorphism with* $p_1 > 0$ *and* $p_n > 0$. *Thus*, $p_i(x,t) \to 0$ *as* $t \to \infty$ *for every* $i \in \tilde{N}$.

We proceed to describe the main results of this paper, which are proved in Sections 2 and 3. See Section 4 for the special case (1.9) and its subcases. The principal results of Section 2, Theorems 1.1 and 1.3, are sufficient conditions for the global loss of a particular allele. Suppose that

(A1) there exist $i \in N$ and constants γ_{ij} such that

$$\gamma_{ij} \geqslant 0, \quad \gamma_{ii} = 0, \quad \sum_j \gamma_{ij} = 1, \tag{1.14a}$$

$$\sum_j \gamma_{ij} r_j(x,p) \geqslant r_i(x,p) \tag{1.14b}$$

for every $x\in\bar{\Omega}$ and every $p\in\Delta$, and that there exists some $x\in\bar{\Omega}$ such that inequality (1.14b) is strict for every $p\in\operatorname{int}\Delta$.

For any $i\in N$ and sufficiently small $\delta>0$, we define

$$\Delta^{(i)}=\{p\in\Delta: p_i=0\}, \tag{1.15a}$$

$$\Delta_\delta^{(i)}=\{p\in\Delta: p_j\geqslant\delta\ \forall j\neq i\}. \tag{1.15b}$$

We have

Theorem 1.1. *If assumption* (A1) *holds, then*

(a) $p_i(x,t)\to 0$ *uniformly in* x *as* $t\to\infty$;

(b) $p(x,0)\in\Delta_\delta^{(i)}$ *implies that*

$$\max_{x\in\bar{\Omega}} p_i(x,t)\leqslant\frac{1}{\delta}\max_{x\in\bar{\Omega}} p_i(x,0) \tag{1.16}$$

for every $t\geqslant 0$.

Part (a) of Theorem 1.1 guarantees the elimination of A_i. Part (b) demonstrates that if $p(x,0)$ is bounded away from every face of the simplex Δ other than $\Delta^{(i)}$, and if $p_i(x,0)$ is small, then $p_i(x,t)$ remains small. Consequently, every equilibrium in $\operatorname{int}\Delta^{(i)}$ that is (asymptotically) stable in $\Delta^{(i)}$ is (asymptotically) stable in Δ.

Remark 1.2. To compare (A1) with (a1), consider the assumption that

(A1*) there exist $i\in N$ and constants γ_{ij} that satisfy (1.14a) and

$$\sum_j \gamma_{ij}r_{jk}(x,p)\geqslant r_{ik}(x,p) \tag{1.17}$$

for every $k\in N$, every $x\in\bar{\Omega}$, and every $p\in\Delta$, and that there exist some $k\in N$ and some $x\in\bar{\Omega}$ such that inequality (1.17) is strict for every $p\in\operatorname{int}\Delta$.

Multiplying (1.17) by p_k, summing over k, and appealing to (1.3), we see that (A1*) implies (A1). The converse is false. Even without frequency dependence, (A1*) is weaker than (a1). Hence, (A1) is much weaker than (a1).

For weak migration, we have more explicit sufficient conditions. We define

$$r^*(x,p)=\max_{j\in N} r_j(x,p) \tag{1.18}$$

and posit that

(A2) there exists $i\in N$ such that $r_i(x,p)<r^*(x,p)$ for every $x\in\bar{\Omega}$ and every $p\in\Delta$;

(A3) for each $k\in N$ and each $x\in\bar{\Omega}$, either $r_k(x,p)=r^*(x,p)$ for every $p\in\Delta$, or $r_k(x,p)<r^*(x,p)$ for every $p\in\Delta$.

Theorem 1.3. *If* (A2) *and* (A3) *hold and* λ *is sufficiently large, then* $p_i(x,t)\to 0$ *uniformly in* x *as* $t\to\infty$.

The intuitive assumption (A2) means that A_i is always less fit than the fittest allele. Assumption (A3) is technical; we expect that it is unnecessary. However, we do not even know whether (A2) alone precludes the existence of an internal equilibrium for sufficiently large λ. For two alleles and $i=1$, assumptions (A2) and (A3) reduce to the requirement that $r_1(x,p)<r_2(x,p)$ for every $x\in\bar{\Omega}$ and every $p\in\Delta$.

The main results of Section 3 are sufficient conditions for protecting a particular allele from loss (Theorem 1.4) and for the existence of an internal equilibrium (Theorem 1.8) and a description of the profile of any internal equilibrium in the zero-migration limit (Theorem 1.9). To present these theorems, we must define some functions.

Let ψ be a positive eigenfunction such that $L_1^\dagger\psi=0$, where $L_1^\dagger$ denotes the adjoint of the closure of L and B (see Sections 2.1 and 4.1 of [17] for details and examples). We normalize $\psi(x)$ so that

$$\int_\Omega \psi(x)\,dx=1. \tag{1.19}$$

For any continuous function $m(x)\not\equiv 0$, consider the linear eigenvalue problem

$$-Lu=\lambda m(x)u \quad \text{in } \Omega, \quad Bu|_{\partial\Omega}=0. \tag{1.20}$$

If m is positive somewhere in $\bar{\Omega}$ and $\int_\Omega m\psi\,dx<0$, then (1.20) has a unique positive eigenvalue $\lambda_1(m)$ with a positive eigenfunction. If $\int_\Omega m\psi\,dx\geqslant 0$, then (1.20) does not have a positive eigenvalue with a positive eigenfunction; in this case, we define $\lambda_1(m)=0$. If $m(x)\leqslant 0$ for every $x\in\bar{\Omega}$, then (1.20) does not have a positive eigenvalue, and we define $\lambda_1(m)=\infty$ [26].

Set

$$F_i(x)=\min_{j\neq i,p\in\Delta}[r_i(x,p)-r_j(x,p)], \tag{1.21a}$$

$$G_i(x)=\min_{j\neq i,k\neq i,p\in\Delta}[r_{ik}(x,p)-r_{jk}(x,p)], \tag{1.21b}$$

$$H_i(x)=\min_{j\neq i,p\in\Delta}[r_{ii}(x,p)-r_{ji}(x,p)]. \tag{1.21c}$$

Theorem 1.4. *If* $F_i\not\equiv 0$ *and* $G_i\not\equiv 0$ *for some* $i\in N$, *then for every* $\lambda>\min[\lambda_1(F_i),\lambda_1(G_i)]$, *there exists* $\delta_i=\delta_i(\lambda)>0$ *such that, for all initial data that satisfy* (1.2), *there exists* t_i, *which may depend on* λ *and the initial data, such that* $p_i(x,t)\geqslant\delta_i$ *for every* $x\in\bar{\Omega}$ *and every* $t\geqslant t_i$.

To start the discussion of the implications of Theorem 1.4, we observe first that if $\max[F_i(x), G_i(x)] \leqslant 0$ for every $x \in \bar{\Omega}$, then Theorem 1.4 cannot ensure protection of the allele A_i for any $\lambda > 0$. Therefore, we now assume that $\max[F_i(x), G_i(x)] > 0$ for some $x \in \bar{\Omega}$. If $F_i(x) > 0$, then A_i is the fittest allele at x; if $G_i(x) > 0$, then A_i is the fittest allele at x when it is rare, i.e., in the limit $p_i \to 0$ (see Proposition B in the appendix).

Remark 1.5. We put

$$\alpha_i = \int_\Omega F_i(x)\psi(x)\,dx, \quad \beta_i = \int_\Omega G_i(x)\psi(x)\,dx. \tag{1.22}$$

There are three cases, each with two subcases.

(1) Suppose that $F_i(x^{(i)}) > 0$ for some $x^{(i)} \in \bar{\Omega}$ and $G_i(x) \leqslant 0$ for every $x \in \bar{\Omega}$. Then A_i is protected (a) for every $\lambda > 0$ if $\alpha_i \geqslant 0$ and (b) for every $\lambda > \lambda_1(F_i)$ if $\alpha_i < 0$.
(2) Suppose that $F_i(x) \leqslant 0$ for every $x \in \bar{\Omega}$ and $G_i(x^{(i)}) > 0$ for some $x^{(i)} \in \bar{\Omega}$. Then A_i is protected (a) for every $\lambda > 0$ if $\beta_i \geqslant 0$ and (b) for every $\lambda > \lambda_1(G_i)$ if $\beta_i < 0$.
(3) Suppose that $F_i(x^{(i)}) > 0$ and $G_i(x^{(i)*}) > 0$ for some $x^{(i)}, x^{(i)*} \in \bar{\Omega}$. Then A_i is protected (a) for every $\lambda > 0$ if $\max(\alpha_i, \beta_i) \geqslant 0$ and (b) for every $\lambda > \min[\lambda_1(F_i), \lambda_1(G_i)]$ if $\max(\alpha_i, \beta_i) < 0$. Thus, if $F_i(x)$ or $G_i(x)$ is positive somewhere, then A_i is protected for sufficiently weak migration.

Remark 1.6. From (1.21) and (1.3) it is easy to see that, even for two alleles, $F_i < G_i$, $F_i = G_i$, and $F_i > G_i$ are all possible. To facilitate application of Theorem 1.4, we note that if $F_i(x) \leqslant G_i(x)$ for every $x \in \bar{\Omega}$, then $\lambda_1(F_i) \geqslant \lambda_1(G_i)$ [26]. In this case, F_i and α_i can be deleted from the above classification: If $G_i(x) \leqslant 0$ for every $x \in \bar{\Omega}$, then Theorem 1.4 cannot ensure protection of A_i for any $\lambda > 0$; if $G_i(x) > 0$ for some $x \in \bar{\Omega}$, then A_i is protected (a) for every $\lambda > 0$ when $\beta_i \geqslant 0$ and (b) for every $\lambda > \lambda_1(G_i)$ when $\beta_i < 0$.

Remark 1.7. Our sufficient condition for protecting A_i requires that the entire face $\Delta^{(i)}$ of the simplex Δ be repelling. Consequently, if $n > 2$, it can be rather stringent.

Finally, we state a theorem on the existence of internal equilibria and one that describes their limiting profiles.

Theorem 1.8. *If $F_i \not\equiv 0$ and $G_i \not\equiv 0$ for every $i \in N$, then for every*

$$\lambda > \lambda_0 = \max_{i \in N} \min[\lambda_1(F_i), \lambda_1(G_i)], \tag{1.23}$$

system (1.8) *has at least one internal equilibrium.*

Set

$$\Omega_i = \{x\in\Omega:\ F_i(x)>0\}\cup\{x\in\Omega:\ G_i(x)>0,\ \ H_i(x)\geqslant 0\}. \tag{1.24}$$

Theorem 1.9. *Suppose that $\Omega_i\neq\emptyset$ for some $i\in N$. If $\hat{p}(x)$ is any internal equilibrium of* (1.8), *then as $\lambda\to\infty$, $\hat{p}_i(x)\to 1$ uniformly in every compact subset of Ω_i.*

If $\max[F_i(x), G_i(x)]\leqslant 0$ for some $i\in N$ and every $x\in\bar{\Omega}$, then Theorem 1.8 cannot guarantee the existence of an internal equilibrium. Therefore, we now assume that for every $i\in N$, there exists $x^{(i)}\in\bar{\Omega}$ such that $\max[F_i(x^{(i)}), G_i(x^{(i)})]>0$. Then an internal equilibrium exists if the allele A_i is protected for every i, as specified in Theorem 1.4 and its discussion. Hence, an internal equilibrium exists for sufficiently weak migration. By Theorem 1.9, as $\lambda\to\infty$, the frequency of each allele converges to 1 in every compact subset of Ω_i. If some alleles are absent at an equilibrium, Theorems 1.8 and 1.9 apply to the subsystem without those alleles.

2. Loss of an allele

In this section, we explore the elimination of a particular allele in system (1.8), where S_i is given by (1.4), under assumption (1.2). We prove Theorems 1.1 and 1.3 in Sections 2.1 and 2.2, respectively.

2.1. Proof of Theorem 1.1

For any positive solution $p(x,t)$ of (1.8) and any constants $\gamma_{ij}\in R^1$, set

$$u_i = \frac{1}{p_i}\prod_{j=1}^{n} p_j^{\gamma_{ij}}. \tag{2.1}$$

Lemma 2.1. *The function u_i satisfies*

$$\begin{aligned}&(\partial_t - L)u_i - (\nabla\ln p_i)\cdot V\nabla u_i\\ &\quad=\left\{\frac{1}{2}\sum_{1\leqslant k<l\leqslant n}\gamma_{ik}\gamma_{il}\left(\nabla\ln\frac{p_k}{p_l}\right)\cdot V\nabla\ln\frac{p_k}{p_l}+\lambda\left[\sum_{k=1}^{n}\gamma_{ik}r_k - r_i\right]\right\}u_i.\end{aligned} \tag{2.2}$$

Proof. For any positive function $w(x,t)$, direct calculation from (1.7a) yields

$$w(\partial_t - L)\ln w = (\partial_t - L)w + \tfrac{1}{2}w(\nabla\ln w)\cdot V\nabla\ln w. \tag{2.3}$$

Choosing $w = p_j$ and recalling (1.8a) and (1.4), we obtain ($j\in N$)

$$(\partial_t - L)\ln p_j = \tfrac{1}{2}(\nabla\ln p_j)\cdot V\nabla\ln p_j + \lambda(r_j - \bar{r}). \tag{2.4}$$

From (2.1), (2.4), and (1.14a) we get

$$\begin{aligned}(\partial_t - L)\ln u_i =& \frac{1}{2}\sum_{k=1}^{n}\gamma_{ik}(\nabla \ln p_k)\cdot V\nabla \ln p_k - \frac{1}{2}(\nabla \ln p_i)\cdot V\nabla \ln p_i \\ &+ \lambda\left[\sum_{k=1}^{n}\gamma_{ik}r_k - r_i\right] \\ =& \frac{1}{2}\sum_{1\leqslant k<l\leqslant n}\gamma_{ik}\gamma_{il}\left(\nabla \ln\frac{p_k}{p_l}\right)\cdot V\nabla \ln\frac{p_k}{p_l} \\ &+ \lambda\left[\sum_{k=1}^{n}\gamma_{ik}r_k - r_i\right] + D, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} D =& -\frac{1}{2}\sum_{1\leqslant k<l\leqslant n}\gamma_{ik}\gamma_{il}\left(\nabla \ln\frac{p_k}{p_l}\right)\cdot V\nabla \ln\frac{p_k}{p_l} \\ &+ \frac{1}{2}\sum_{k=1}^{n}\gamma_{ik}(\nabla \ln p_k)\cdot V\nabla \ln p_k - \frac{1}{2}(\nabla \ln p_i)\cdot V\nabla \ln p_i \\ =& \frac{1}{2}(\nabla \ln u_i)\cdot V\nabla \ln u_i + (\nabla \ln p_i)\cdot V\nabla \ln u_i. \end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5) and the result into (2.3) with $w = u_i$ leads to (2.2), which proves Lemma 2.1. □

Lemma 2.2. *Suppose that assumption* (A1) *holds. Then* $\min_{x\in\bar{\Omega}} u_i(x,t)$ *is strictly increasing for* $t>0$.

Proof. We argue by contradiction: suppose that $\min_{x\in\bar{\Omega}} u_i(x,t^*)\geqslant \min_{x\in\bar{\Omega}} u_i(x,t^{**})$ for some $0\leqslant t^*<t^{**}$. Then there exist some $\bar{x}\in\bar{\Omega}$ and $\bar{t}\in(t^*,t^{**}]$ such that

$$u_i(\bar{x},\bar{t}) = \min_{x\in\bar{\Omega},\ t^*\leqslant t\leqslant t^{**}} u_i(x,t). \tag{2.7}$$

Lemma 2.1 informs us that u_i satisfies (2.2). By (1.14) and the positive definiteness of V, the right-hand side of (2.2) is non-negative. Moreover, $Bu_i = 0$ on $\partial\Omega\times(0,\infty)$. Therefore, by the maximum principle [23, Chapter 3] we see that $u_i(x,t)\equiv u_i(\bar{x},\bar{t})>0$ for every $x\in\bar{\Omega}$ and $t^*\leqslant t\leqslant\bar{t}$. Furthermore,

$$\sum_{k=1}^{n}\gamma_{ik}r_k - r_i \equiv 0 \tag{2.8}$$

for every $x\in\bar{\Omega}$ and $t^*\leqslant t\leqslant\bar{t}$, which contradicts (A1). □

Proof of Theorem 1.1. We first prove part (a). Pick any sequence $\{t_k^*\}_{k=1}^{\infty}$ such that $t_k^* \to \infty$ as $k \to \infty$. Since $0 \leqslant p_j \leqslant 1$, the estimate $(j \in N)$

$$\sup_{t \geqslant \varepsilon} ||p_j(\cdot, t)||_{C^{2,r}(\bar{\Omega})} \leqslant C_1 < \infty, \tag{2.9}$$

where $r \in (0,1)$ and $\varepsilon > 0$, is well known (see [24]). Passing to a subsequence if necessary, we may assume that $p(x, t_k^*) \to p^*(x)$ as $k \to \infty$. We claim that $p_i^*(x) \equiv 0$. Suppose not, i.e., $p_i^*(x) \geqslant 0$ and $p_i^* \not\equiv 0$.

We first consider the case $p_i^*(x) > 0$ in $\bar{\Omega}$ and derive a contradiction. Since $p_i(x, t_k^*) \to p_i^*(x) > 0$, we see from (2.1) that the sequence $\{\min_{x \in \bar{\Omega}} u_i(x, t_k^*)\}_{k=1}^{\infty}$ is bounded above. By Lemma 2.2, $\min_{x \in \bar{\Omega}} u_i(x,t)$ is increasing. Therefore,

$$\lim_{t \to \infty} \min_{x \in \bar{\Omega}} u_i(x,t) = u_i^* > 0. \tag{2.10}$$

Set

$$p^{(k)}(x,t) = p(x, t + t_k^*), \quad k \geqslant 1. \tag{2.11}$$

By (2.9) and regularity theory for linear parabolic operators, we have $(j \in N)$

$$||p_j^{(k)}||_{C^{2+r, 1+\frac{r}{2}}(\bar{\Omega} \times [\varepsilon, \infty))} \leqslant C_2 < \infty, \tag{2.12}$$

where C_2 is independent of k. By (2.12), the Arzela–Ascoli lemma, and a standard diagonal process, passing to a subsequence if necessary, we may assume that $p^{(k)}(x,t) \to \tilde{p}(x,t)$ in $C^{2,1}(\bar{\Omega} \times [\varepsilon, T])$ for every $0 < \varepsilon < T < \infty$ as $k \to \infty$, which together with (1.8) and (2.11) implies that $\tilde{p}(x,t)$ satisfies

$$\tilde{p}_{j,t} = L\tilde{p}_j + S_j(x, \tilde{p}) \quad \text{in } \Omega \times (0, \infty), \tag{2.13a}$$

$$B\tilde{p}_j = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{2.13b}$$

$$\tilde{p}_j(x,0) = p_j^*(x), \quad \sum_{j=1}^{n} \tilde{p}_j(x,0) \equiv 1 \quad \text{in } \Omega. \tag{2.13c}$$

Since $\tilde{p}_i(x,0) = p_i^*(x) > 0$, the maximum principle informs us that $\tilde{p}_i(x,t) > 0$ for all $t > 0$. Since $\min_{x \in \bar{\Omega}} u_i(x,t)$ is increasing and $p_i^*(x) > 0$, therefore $p_j^*(x)$ is positive for every j such that $\gamma_{ij} > 0$. Therefore, $\tilde{p}_j(x,t) > 0$ if $\gamma_{ij} > 0$, and we can define $\tilde{u}_i(x,t)$ by

$$\tilde{u}_i = \frac{1}{\tilde{p}_i} \prod_{j=1}^{n} \tilde{p}_j^{\gamma_{ij}}, \tag{2.14}$$

where we apply the convention that $\tilde{p}_j^{\gamma_{ij}} = 1$ if $\gamma_{ij} = 0$. Hence, $\tilde{u}_i(x,0) > 0$. Lemma 2.2 shows that $\min_{x \in \bar{\Omega}} \tilde{u}_i(x,t)$ is strictly increasing. In particular, there exists $\eta > 0$ such that

$$\min_{x \in \bar{\Omega}} \tilde{u}_i(x,1) \geqslant \min_{x \in \bar{\Omega}} \tilde{u}_i(x,0) + \eta = u_i^* + \eta. \tag{2.15}$$

On the one hand, since $u_i(x,1+t_k^*)$ converges uniformly to $\tilde{u}_i(x,1)$ as $k\to\infty$, we see from (2.15) that

$$\min_{x\in\bar{\Omega}} u_i(x,1+t_K^*)\geqslant u_i^*+\tfrac{1}{2}\eta \tag{2.16}$$

for some sufficiently large K. On the other hand, since $\min_{x\in\bar{\Omega}}u_i(x,t)$ is increasing, we find $u_i^*\geqslant \min_{x\in\bar{\Omega}} u_i(x,1+t_K^*)$, which contradicts (2.16). Therefore, the case $p_i^*(x)>0$ in $\bar{\Omega}$ cannot occur.

If $p_i^*(x)\geqslant 0$ and $p_i^*\not\equiv 0$, the maximum principle and (2.13) imply that $\tilde{p}_i(x,t)>0$ for $t>0$. Choose $\delta^*>0$ and set $\check{p}(x,t)=\tilde{p}(x,t+\delta^*)$. Note that $\check{p}(x,t)$ also satisfies (2.13) with $p_j^*(x)$ replaced by $\tilde{p}_j(x,\delta^*)$. Since $\check{p}_i(x,0)>0$, we can argue as above to reach a contradiction. Therefore, passing to a subsequence if necessary, we can conclude that $p_i(x,t_k^*)\to 0$ as $t_k^*\to\infty$.

Since the sequence $\{t_k^*\}_{k=1}^{\infty}$ is chosen arbitrarily and $\sup_{t\geqslant\varepsilon}||p_i(\cdot,t)||_{C^{2,r}(\bar{\Omega})}<\infty$ for every $\varepsilon>0$, a standard compactness argument shows that $p_i(x,t)\to 0$ uniformly as $t\to\infty$. This proves part (a) of Theorem 1.1.

For the proof of part (b), by Lemma 2.2 we have

$$\max_{x\in\bar{\Omega}}\frac{p_i(x,t)}{\prod_{j=1}^n p_j^{\gamma_{ij}}(x,t)}\leqslant \max_{x\in\bar{\Omega}}\frac{p_i(x,0)}{\prod_{j=1}^n p_j^{\gamma_{ij}}(x,0)} \tag{2.17}$$

for every $t\geqslant 0$. Since $p(x,t)\in\Delta$ and $p(x,0)\in\Delta_\delta^{(i)}$, by (2.17) we have

$$\max_{x\in\bar{\Omega}} p_i(x,t)\leqslant \max_{x\in\bar{\Omega}}\frac{p_i(x,t)}{\prod_{j=1}^n p_j^{\gamma_{ij}}(x,t)}\leqslant\frac{\max_{x\in\bar{\Omega}} p_i(x,0)}{\prod_{j\neq i}\min_{x\in\bar{\Omega}} p_j^{\gamma_{ij}}(x,0)}\leqslant\frac{1}{\delta}\max_{x\in\bar{\Omega}} p_i(x,0). \tag{2.18}$$

2.2. Proof of Theorem 1.3

By (A2) and (A3) we see that for every $x\in\bar{\Omega}$, there exists a nonempty proper subset of N, denoted by N_x, such that

$$r_k(x,p)=r^*(x,p)\quad \forall k\in N_x,\ \forall p\in\Delta, \tag{2.19a}$$

$$r_k(x,p)<r^*(x,p)\quad \forall k\in N-N_x,\ \forall p\in\Delta. \tag{2.19b}$$

Now set

$$\bar{p}(y,t)=\sum_{k\in N_x} p_k(y,t),\quad y\in\bar{\Omega},\ t\geqslant 0, \tag{2.20a}$$

$$h(y)=\min_{p\in\Delta}\left[\min_{k\in N_x} r_k(y,p)-\max_{k\in N-N_x} r_k(y,p)\right],\quad y\in\bar{\Omega}. \tag{2.20b}$$

Let $p^*(y,t)$ be the solution of

$$p_t^* = Lp^* + \lambda h(y)p^*(1-p^*) \quad \text{in } \Omega\times(0,\infty), \tag{2.21a}$$

$$Bp^* = 0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{2.21b}$$

$$p^*(y,0) = \bar{p}(y,0). \tag{2.21c}$$

Lemma 2.3. *Suppose that* (A2) *and* (A3) *hold. Then* $\bar{p}(y,t)\geqslant p^*(y,t)$ *for every* $y\in\bar{\Omega}$ *and every* $t\geqslant 0$.

Proof. By (1.8a) we have

$$\begin{aligned}
\bar{p}_t - L\bar{p} &= \lambda \sum_{k\in N_x} p_k(r_k-\bar{r}) \\
&= \lambda \sum_{k\in N_x} p_k\left[r_k - \sum_{j\in N_x} p_j r_j - \sum_{j\in N-N_x} p_j r_j\right] \\
&= \lambda \sum_{j\in N_x} p_j r_j\left(1-\sum_{k\in N_x} p_k\right) - \lambda\left(\sum_{k\in N_x} p_k\right)\sum_{j\in N-N_x} p_j r_j \\
&\geqslant \lambda(1-\bar{p})\sum_{j\in N_x} p_j r_j - \lambda\bar{p}(1-\bar{p})\max_{j\in N-N_x} r_j \\
&= \lambda(1-\bar{p})\left[\sum_{j\in N_x} p_j r_j - \bar{p}\max_{j\in N-N_x} r_j\right] \\
&\geqslant \lambda\bar{p}(1-\bar{p})\left[\min_{j\in N_x} r_j - \max_{j\in N-N_x} r_j\right] \\
&\geqslant \lambda h(y)\bar{p}(1-\bar{p}).
\end{aligned} \tag{2.22}$$

Since $\bar{p}$ also satisfies boundary condition (2.21b) and the initial condition (2.21c), by the comparison principle [23, Chapter 3] we have $\bar{p}(y,t)\geqslant p^*(y,t)$ for every $y\in\bar{\Omega}$ and every $t\geqslant 0$. This proves Lemma 2.3. □

Lemma 2.4. *For every* $x\in\bar{\Omega}$, *there exist positive constants* $a=a(x)$, $\hat{\delta}=\hat{\delta}(x)$, *and* $\Lambda=\Lambda(x)$ *such that for every* $\lambda\geqslant\Lambda$, *there exists some positive constant* $\tau=\tau(x,\lambda)$ *such that*

$$r_i(y,p(y,t)) - \bar{r}(y,p(y,t)) \leqslant -\hat{\delta} \tag{2.23}$$

for every $y\in B_a(x)\cap\Omega$ *and every* $t\geqslant\tau$, *where* $B_a(x)$ *is the Euclidean ball in* R^d *centered at* x *with radius* a.

Proof. By (A2) we see that $h(x)>0$. Hence, there exists some positive constant $a=a(x)$ such that $h(y)\geqslant\frac{1}{2}h(x)>0$ for every $y\in B_{2a}(x)\cap\Omega$. For a later purpose, we

choose a so small that

$$r_k(y,p) \geqslant r^*(y,p) - \tfrac{1}{4}C_3, \tag{2.24}$$

where

$$C_3 = C_3(x) = \min_{y \in B_{2a}(x) \cap \bar{\Omega},\, p \in \Delta} [r^*(y,p) - r_i(y,p)], \tag{2.25}$$

for every $k \in N_x$, every $y \in B_{2a}(x) \cap \Omega$, and every $p \in \Delta$. By (A2) we see that $C_3 > 0$. Therefore, (2.24) follows from (2.19) and the continuity of functions r_k and r^*.

By Theorem 2.1 of [17] we see that $p^*(y,t) \to p^\infty(y)$ uniformly in y as $t \to \infty$, where $p^\infty(y)$ is an equilibrium of (2.21). There are two cases:

(a) $h(y) \geqslant 0$ for every $y \in \bar{\Omega}$, in which case $p^\infty \equiv 1$.

(b) h changes sign in $\bar{\Omega}$, in which case, if λ is sufficiently large, $p^\infty(y)$ is the unique solution of the problem

$$Lp^\infty + \lambda h(y) p^\infty (1 - p^\infty) = 0 \quad \text{in } \Omega,\ 0 < p^\infty < 1 \quad \text{in } \bar{\Omega},\ Bp^\infty = 0 \text{ on } \partial\Omega. \tag{2.26}$$

As $\lambda \to \infty$, $p^\infty \to 1$ uniformly in any compact subset of $\{y \in \bar{\Omega} : h(y) > 0\}$. Since $h(y) > 0$ in $\overline{B_{2a}(x) \cap \Omega}$, we see that for any $\varepsilon > 0$, there exists $\Lambda^* = \Lambda^*(x,\varepsilon) \gg 1$ such that if $\lambda \geqslant \Lambda^*$, then

$$p^\infty(y) \geqslant 1 - \varepsilon \quad \forall y \in B_a(x) \cap \Omega. \tag{2.27}$$

Since $p^* \to p^\infty$ uniformly as $t \to \infty$, there exists $\tau^* = \tau^*(x,\lambda,\varepsilon) \gg 1$ such that

$$|p^*(y,t) - p^\infty(y)| \leqslant \varepsilon \quad \forall y \in \bar{\Omega},\ \forall t \geqslant \tau^*. \tag{2.28}$$

By (2.27), (2.28), and Lemma 2.3 we have

$$\bar{p}(y,t) \geqslant p^*(y,t) \geqslant p^\infty(y) - \varepsilon \geqslant 1 - 2\varepsilon \quad \forall y \in B_a(x) \cap \Omega,\ \forall t \geqslant \tau^*. \tag{2.29}$$

Set

$$C_4 = \max_{y \in \bar{\Omega},\, p \in \Delta,\, k \in N} |r_k(y,p)| > 0. \tag{2.30}$$

We now assume that $y \in B_a(x) \cap \Omega$ and $t \geqslant \tau^*$, and estimate $r_i - \bar{r}$ by writing

$$\begin{aligned} r_i(y,p(y,t)) - \bar{r}(y,p(y,t)) &= r_i - r^* + \sum_{k \in N} p_k (r^* - r_k) \\ &= D_1 + D_2 + D_3 + D_4, \end{aligned} \tag{2.31}$$

where (2.25), (2.24), (2.20a), and (2.30) yield

$$D_1 = r_i(y, p(y,t)) - r^*(y, p(y,t)) \leqslant -C_3, \tag{2.32a}$$

$$D_2 = \sum_{k \in N_x} (r^* - r_k) p_k \leqslant \sum_{k \in N_x} \frac{1}{4} C_3 p_k \leqslant \frac{1}{4} C_3, \tag{2.32b}$$

$$D_3 = r^*(1 - \bar{p}) \leqslant 2C_4 \varepsilon, \tag{2.32c}$$

$$D_4 = -\sum_{k \in N - N_x} r_k p_k \leqslant C_4 \sum_{k \in N - N_x} p_k = C_4(1 - \bar{p}) \leqslant 2C_4 \varepsilon. \tag{2.32d}$$

By choosing $\varepsilon = \varepsilon(x) = C_3(x)/(16C_4)$ and $\hat{\delta} = \frac{1}{2} C_3(x)$, we see that (2.23) holds for $\lambda \geqslant \Lambda^*(x, \varepsilon(x)) \equiv \Lambda(x)$, $y \in B_a(x) \cap \Omega$, and $t \geqslant \tau^*(x, \lambda, \varepsilon(x)) \equiv \tau(x, \lambda)$. This proves Lemma 2.4. □

Proof of Theorem 1.3. Since $\bar{\Omega} \subset \bigcup_{x \in \bar{\Omega}} B_a(x)$, therefore, by the compactness of $\bar{\Omega}$, there exist finitely many points, denoted by $x^{(1)}, \ldots, x^{(l)}$, such that $\bar{\Omega} \subset \bigcup_{j=1}^{l} B_{a(x^{(j)})}(x^{(j)})$. Now set $\bar{\delta} = \min_{1 \leqslant j \leqslant l} \hat{\delta}(x^{(j)})$ and $\bar{\Lambda} = \max_{1 \leqslant j \leqslant l} \Lambda(x^{(j)})$, and for every $\lambda \geqslant \Lambda$, let $\bar{\tau} = \max_{1 \leqslant j \leqslant l} \tau(x^{(j)}, \lambda)$.

By Lemma 2.4 we see that for $\lambda \geqslant \bar{\Lambda}$,

$$r_i(y, p(y,t)) - \bar{r}(y, p(y,t)) \leqslant -\bar{\delta} \tag{2.33}$$

for every $y \in \bar{\Omega}$ and every $t \geqslant \bar{\tau}$. Therefore, (1.8a) and (1.4) imply that

$$p_{i,t} \leqslant Lp_i - \lambda \bar{\delta} p_i \tag{2.34}$$

for every $y \in \Omega$ and every $t \geqslant \bar{\tau}$. Hence, if $\lambda \geqslant \bar{\Lambda}$, by the comparison principle we have

$$p_i(y,t) \leqslant \max_{y \in \bar{\Omega}} p_i(y, \bar{\tau}) \cdot e^{-\lambda \bar{\delta}(t - \bar{\tau})} \leqslant e^{-\lambda \bar{\delta}(t - \bar{\tau})} \tag{2.35}$$

for every $y \in \bar{\Omega}$ and every $t \geqslant \bar{\tau}$. In particular, $p_i(x,t) \to 0$ uniformly in x as $t \to \infty$. □

3. Protection of alleles

In Section 3.1, we prove Theorem 1.4 on protection of a particular allele from loss. We devote Section 3.2 to the proof of Theorem 1.8, which states that if every allele is protected, then there exists at least one internal equilibrium. In Section 3.3, we establish Theorem 1.9 on the profile of any internal equilibrium in the zero-migration limit.

3.1. Proof of Theorem 1.4

First, we study the case when G_i is positive somewhere in Ω, and postpone to the end of this subsection the discussion of the case when F_i is positive somewhere in Ω.

Let $q(x,t)$ be the solution of

$$q_t = Lq + \lambda q(1-q)[G_i(x)(1-q) + H_i(x)q] \quad \text{in } \Omega \times (0,\infty), \tag{3.1a}$$

$$Bq = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{3.1b}$$

$$q(x,0) = p_i(x,0). \tag{3.1c}$$

Lemma 3.1. *The inequality $p_i(x,t) \geqslant q(x,t)$ holds for every $x \in \bar{\Omega}$ and every $t \geqslant 0$.*

Proof. We claim that $p_i(x,t)$ satisfies

$$p_{i,t} \geqslant Lp_i + \lambda p_i(1-p_i)[G_i(x)(1-p_i) + H_i(x)p_i] \quad \text{in } \Omega \times (0,\infty). \tag{3.2}$$

To establish this assertion, it suffices to show that

$$\begin{aligned} r_i(x,p) - \bar{r}(x,p) &= \sum_{j\neq i} [r_i(x,p) - r_j(x,p)]p_j \\ &= \sum_{j\neq i} \left[\sum_{k\neq i} (r_{ik} - r_{jk})p_k + (r_{ii} - r_{ji})p_i \right] p_j \\ &\geqslant \sum_{j\neq i} \left[G_i(x) \sum_{k\neq i} p_k + H_i(x)p_i \right] p_j \\ &= (1-p_i)[G_i(x)(1-p_i) + H_i(x)p_i]. \end{aligned} \tag{3.3}$$

By the comparison principle we see that $p_i(x,t) \geqslant q(x,t)$ for every $x \in \bar{\Omega}$ and every $t \geqslant 0$, which proves Lemma 3.1. □

Recall the discussion of (1.20) and consider the linear eigenvalue problem

$$L\phi + \lambda m(x)\phi = -\mu\phi \quad \text{in } \Omega, \quad B\phi|_{\partial\Omega} = 0. \tag{3.4}$$

Denote the principal eigenvalue of (3.4) by $\mu_1 = \mu_1(\lambda)$ and the corresponding eigenfunction by ϕ_1. By the Krein–Rutman Theorem [15] and the maximum principle, we may assume that $\phi_1 > 0$ in $\bar{\Omega}$. We normalize ϕ_1 so that $\max_{\bar{\Omega}} \phi_1 = 1$. The following result is well known (see Senn [25], where $\gamma(\lambda)$ is equivalent to our $-\mu_1(\lambda)$).

Lemma 3.2. *Suppose that $m(x)$ is positive somewhere. Then $\mu_1(\lambda) < 0$ for every $\lambda > \lambda_1(m)$.*

Next we have

Lemma 3.3. *Suppose that G_i is positive somewhere. Then for every $\lambda>\lambda_1(G_i)$, there exists $\tilde{\delta}=\tilde{\delta}(\lambda)>0$ such that for any solution $\hat{q}$ of the problem*

$$L\hat{q}+\lambda\hat{q}(1-\hat{q})[G_i(x)(1-\hat{q})+H_i(x)\hat{q}]=0 \quad \text{in } \Omega, \quad B\hat{q}|_{\partial\Omega}=0, \quad 0<\hat{q}(x)\leqslant 1, \tag{3.5}$$

we have $\hat{q}(x)\geqslant\tilde{\delta}$ for every $x\in\bar{\Omega}$.

Proof. We argue by contradiction. Suppose that the conclusion of Lemma 3.3 fails for some $\lambda>0$. Then there exists a sequence of functions $\{q_k(x)\}_{k=1}^{\infty}$ that are solutions of (3.5) such that $\min_{\bar{\Omega}}q_k\to 0$ as $k\to\infty$. By the global Harnack inequality [7, Chapter 7; 16], there exists some positive constant C, independent of k, such that $\max_{\bar{\Omega}}q_k\leqslant C\min_{\bar{\Omega}}q_k$. Therefore, $\max_{\bar{\Omega}}q_k\to 0$ as $k\to\infty$. Setting $\tilde{q}_k=q_k/||q_k||_\infty$, from (3.5) we obtain

$$L\tilde{q}_k+\lambda\tilde{q}_k(1-q_k)[G_i(1-q_k)+H_iq_k]=0 \quad \text{in } \Omega, \tag{3.6a}$$

$$B\tilde{q}_k|_{\partial\Omega}=0, \quad \tilde{q}_k(x)>0, \ ||\tilde{q}_k||_\infty=1. \tag{3.6b}$$

By elliptic regularity [7, Chapters 6 and 9] and the Sobolev embedding theorem [7, p. 158], passing to a subsequence if necessary, we may assume that $\tilde{q}_k\to q^*$ in $C^2(\bar{\Omega})$ and that q^* satisfies

$$Lq^*+\lambda G_i(x)q^*=0 \quad \text{in } \Omega, \quad Bq^*|_{\partial\Omega}=0, \quad q^*(x)\geqslant 0, \quad ||q^*||_\infty=1. \tag{3.7}$$

Hence, $\mu_1(\lambda)=0$ for some $\lambda>0$, which contradicts Lemma 3.2. □

Proof of Theorem 1.4. We first consider the case where G_i is positive somewhere. Choose $m(x)=G_i(x)$ in (3.4) and set $\underline{q}(x)=\check{\delta}\phi_1(x)$. First choose $\check{\delta}>0$ so small that $\underline{q}(x)\leqslant p_i(x,0)$ for every $x\in\bar{\Omega}$; this is possible because $p_i(x,0)>0$ in $\bar{\Omega}$. For sufficiently small $\check{\delta}$, from (3.4) we get

$$\begin{aligned} &L\underline{q}+\lambda\underline{q}(1-\underline{q})[G_i(1-\underline{q})+H_i\underline{q}] \\ &\quad=\check{\delta}\phi_1\{-\mu_1+\lambda\check{\delta}\phi_1[-G_i+(1-\check{\delta}\phi_1)(H_i-G_i)]\}>0 \end{aligned} \tag{3.8}$$

in Ω, since $\mu_1(\lambda)<0$ for every $\lambda>\lambda_1(G_i)$. Thus,

$$0=\underline{q}_t<L\underline{q}+\lambda\underline{q}(1-\underline{q})[G_i(x)(1-\underline{q})+H_i(x)\underline{q}] \quad \text{in } \Omega\times(0,\infty). \tag{3.9}$$

Now consider the parabolic problem

$$\check{q}_t=L\check{q}+\lambda\check{q}(1-\check{q})[G_i(x)(1-\check{q})+H_i(x)\check{q}] \quad \text{in } \Omega\times(0,\infty), \tag{3.10a}$$

$$B\check{q}=0 \quad \text{on } \partial\Omega\times(0,\infty), \tag{3.10b}$$

$$\check{q}(x,0) = \check{\delta}\phi_1(x). \tag{3.10c}$$

Since $0<\check{\delta}\phi_1(x)<1$, it follows from the comparison principle that $0<\check{q}(x,t)<1$ for every $x\in\bar{\Omega}$ and every $t\geqslant 0$. Since $\underline{q}(x)$ is a subsolution of (3.10), it also follows from the comparison principle that for every $x\in\bar{\Omega}$, $\check{q}(x,t)$ is monotone increasing in t. Hence, the limit $Q(x)=\lim_{t\to\infty}\check{q}(x,t)$ exists for every x, and it satisfies $0<Q(x)\leqslant 1$. Moreover, by elliptic regularity we see that $Q(x)$ is a classical solution of (3.7). Hence, by Lemma 3.3 we have $Q(x)\geqslant\tilde{\delta}$ in $\bar{\Omega}$. Set $\delta^*=\frac{1}{2}\tilde{\delta}$. Since $\check{q}(x,t)\to Q(x)$ pointwise, by standard parabolic regularity and compactness arguments we see that $\check{q}(x,t)\to Q(x)$ uniformly as $t\to\infty$. Therefore, there exists $t_i>0$ such that $\check{q}(x,t)\geqslant\frac{1}{2}Q(x)\geqslant\delta^*$ for every $x\in\bar{\Omega}$ and $t\geqslant t_i$. By the comparison principle we have $q(x,t)\geqslant\check{q}(x,t)$ for every $x\in\bar{\Omega}$ and $t\geqslant 0$. Therefore, it follows from Lemma 3.1 that $p_i(x,t)\geqslant q(x,t)\geqslant\check{q}(x,t)\geqslant\delta^*$ for every $x\in\bar{\Omega}$ and $t\geqslant t_i$. This proves Theorem 1.4 for the case where G_i is positive somewhere and $\lambda>\lambda_1(G_i)$.

For the case where F_i is positive somewhere and $\lambda>\lambda_1(F_i)$, it suffices to replace both $G_i(x)$ and $H_i(x)$ by $F_i(x)$ between (3.1) and (3.10). To check that (3.2) still holds with this substitution, it suffices to observe that, instead of (3.3), we have

$$r_i(x,p)-\bar{r}(x,p)=\sum_{j\neq i}[r_i(x,p)-r_j(x,p)]p_j\geqslant F_i(x)\sum_{j\neq i}p_j=F_i(x)(1-p_i). \tag{3.11}$$

3.2. *Proof of Theorem 1.8*

We first convert (1.8) into a semi-dynamical system on a suitable function space. Set

$$Y=\{p\in\{C(\bar{\Omega})\}^n\colon\ p(x)\in\Delta\ \forall x\in\bar{\Omega}\}, \tag{3.12a}$$

$$Y_0=\{p\in Y\colon\ p_i(x)>0\ \forall i\in N,\ \forall x\in\bar{\Omega}\}. \tag{3.12b}$$

For $t>0$, let $\pi(p^0,t)$ denote the unique solution $p(x,t)$ of (1.8) such that $p(x,0)=p^0(x)$. Clearly, $\pi(p^0,t)$ exists for all $t>0$ for every $p^0(x)\in\Delta$. By the theory of analytical semigroups and parabolic differential equations [9, Chapter 3], we see that $\pi(p^0,t)$ is a semi-dynamical system on Y equipped with the standard supremum-norm metric $d_0(p,q)=\sum_i||p_i-q_i||_\infty$.

Definition 3.4. The semi-dynamical system π is permanent if there is a subset U of Y_0 with the following properties: (i) U is bounded; (ii) $\inf_{p\in U}d_0(p,Y-Y_0)>0$; (iii) for any $p^0\in Y_0$, $d_0(\pi(p^0,t),U)\to 0$ as $t\to\infty$.

For results on permanence for reaction–diffusion models, see [1, Chapter 4]; [2–4,6,8,10–13] and the references therein. From Theorem 4.6 of Cantrell and Cosner [1, p. 241], we have

Theorem 3.5. *If π is permanent, then* (1.8) *has an internal equilibrium.*

Proof of Theorem 1.8. We show that π is permanent when $\lambda>\lambda_0$, i.e., $\lambda>\min[\lambda_1(F_i),\lambda_1(G_i)]$ for every $i\in N$. For any $\lambda>\lambda_0$, set $\hat{\delta}=\min_{i\in N}\delta_i$, where δ_i is defined in Theorem 1.4. Let

$$U=\{p\in Y_0:\ p_i(x)\geqslant\hat{\delta}\ \forall i\in N,\ \forall\ x\in\bar{\Omega}\}. \tag{3.13}$$

Clearly, U is bounded, and there exists some positive constant c_n, which depends only on n, such that

$$d_0(p,\ Y-\ Y_0)\geqslant c_n\hat{\delta}\quad\forall p\in U. \tag{3.14}$$

By Theorem 1.4, for any initial data p^0, we have $p_i(x,t)\geqslant\hat{\delta}$ for every $i\in N$, every $x\in\bar{\Omega}$, and every $t\geqslant T^*=\max_{i\in N}t_i$, where t_i is given in Theorem 1.4. That is, $p(\cdot,t)\in U$ if $t\geqslant T^*$. Hence, Theorem 3.5 applies, and this proves that (1.8) has at least an internal equilibrium when $\lambda>\lambda_0$. □

3.3. Proof of Theorem 1.9

We first consider the subdomain $\Omega_i^*=\{x\in\Omega:\ G_i(x)>0,\ H_i(x)\geqslant 0\}$ of Ω_i. For any compact set $K\subset\Omega_i^*$, we can always find an open set $\tilde{\Omega}$ such that $K\subset\subset\tilde{\Omega}\subset\subset\Omega_i^*$. Define $\eta=\inf_{\tilde{\Omega}}G_i$. Obviously, $\eta>0$. As in (3.2), the equilibrium $\hat{p}(x)$ satisfies

$$-L\hat{p}_i\geqslant\lambda\hat{p}_i(1-\hat{p}_i)[G_i(x)(1-\hat{p}_i)+H_i(x)\hat{p}_i]\quad\text{in }\Omega. \tag{3.15}$$

Since $G_i(x)\geqslant\eta$ and $H_i(x)\geqslant 0$ in $\tilde{\Omega}$, we have

$$-L\hat{p}_i\geqslant\lambda\eta\hat{p}_i(1-\hat{p}_i)^2\quad\text{in }\tilde{\Omega}. \tag{3.16}$$

Therefore, $\hat{p}_i$ is a supersolution of the problem

$$Lw+\lambda\eta w(1-w)^2=0\quad\text{in }\tilde{\Omega},\ \ w|_{\partial\tilde{\Omega}}=0,\ \ w>0\quad\text{in }\tilde{\Omega}. \tag{3.17}$$

Next, we show that (3.17) has a unique solution if $\lambda>\lambda_1^{\mathrm{D}}(\tilde{\Omega})$, where $\lambda_1^{\mathrm{D}}(\tilde{\Omega})$ is the principal eigenvalue of L with respect to zero-Dirichlet boundary conditions. One way to prove this assertion is as follows: By the maximum principle we see that any positive solution w of (3.17) satisfies $w\leqslant 1$. For the existence of w, note that $\bar{w}\equiv 1$ is a supersolution, and $\underline{w}=\varepsilon\psi_1$ is a subsolution for any sufficiently small $\varepsilon>0$, where $\psi_1>0$ is an eigenfunction corresponding to $\lambda_1^{\mathrm{D}}(\tilde{\Omega})$. The uniqueness of w follows from a supersolution-subsolution argument (see [22, Chapter 3]) and the monotonicity of $f(w)/w$ for the nonlinearity $f(w)=w(1-w)^2$.

Since $\hat{p}_i$ is a supersolution of (3.17) and $\underline{w}$ is a subsolution, by the supersolution–subsolution method and the uniqueness of the problem we have $\hat{p}_i\geqslant w$

in $\tilde{\Omega}$. It is known that for the unique solution w of (3.17), as $\lambda\to\infty$, $w\to 1$ uniformly in any compact subset of $\tilde{\Omega}$. In particular, this implies that $\hat{p}_i\to 1$ uniformly in K.

For the subdomain $\Omega_i^{**}=\{x\in\Omega\colon F_i(x)>0\}$, as in (3.11), we have

$$-L\hat{p}_i\geqslant\lambda F_i(x)\hat{p}_i(1-\hat{p}_i)\geqslant\lambda F_i(x)\hat{p}_i(1-\hat{p}_i)^2 \quad \text{in } \Omega_i^{**}. \tag{3.18}$$

The rest of the proof is exactly the same as the previous case. This completes the proof of Theorem 1.9. □

4. Frequency-independent selection

In this section, we posit that the selection coefficients depend only on position, i.e., (1.9) holds. This natural assumption, which is made in most of the literature, enables us to simplify and amplify our results without losing their qualitative essence. In Section 4.1, we present our general results, which we specialize in the following three subsections. In Sections 4.2–4.4, we treat selection coefficients without dominance, multiplicative selection coefficients, and genotype-independent spatial dependence, respectively.

4.1. General results

Assumption (1.9) does not simplify Theorems 1.1 and 1.3. We now use (1.9) to simplify Theorems 1.4, 1.8, and 1.9.

In Remark 1.6, we simplified Theorem 1.4 under the assumption that $F_i(x)\leqslant G_i(x)$ for every $x\in\bar{\Omega}$, which is precisely the conclusion of

Proposition 4.1. *If* (1.9) *holds, then* $F_i(x)\leqslant G_i(x)$ *for every* $i\in N$ *and every* $x\in\bar{\Omega}$.

Proof. From (1.21) and (1.3) we have

$$\begin{aligned}
F_i(x)&=\min_{j\neq i,p\in\Delta}\sum_k[r_{ik}(x)-r_{jk}(x)]p_k\\
&=\min_{j\neq i,k\in N}[r_{ik}(x)-r_{jk}(x)]\\
&=\min\left\{\min_{j\neq i,k\neq i}[r_{ik}(x)-r_{jk}(x)],\min_{j\neq i}[r_{ii}(x)-r_{ji}(x)]\right\}\\
&=\min[G_i(x),H_i(x)]\leqslant G_i(x).
\end{aligned}\tag{4.1}$$

Remark 4.2. If A_i is recessive to A_l for some $l \neq i$, then

$$G_i(x) \leqslant r_{il}(x) - r_{ll}(x) = 0 \tag{4.2}$$

for every $x \in \bar{\Omega}$, so Remark 1.6 cannot guarantee the protection of A_i.

Remark 4.3. If there are only two alleles, from (1.21b) we obtain

$$G_1(x) = r_{12}(x) - r_{22}(x), \quad G_2(x) = r_{12}(x) - r_{11}(x). \tag{4.3}$$

Hence, if there is overdominance in the entire habitat, i.e.,

$$r_{12}(x) > \max[r_{11}(x), r_{22}(x)] \tag{4.4}$$

for every $x \in \bar{\Omega}$, then both A_1 and A_2 are protected.

Remark 4.4. Remark 1.6 and Proposition 4.1 demonstrate that F_i can be omitted from Theorems 1.8 and 1.9, which apply with

$$\lambda_0 = \max_{i \in N} \lambda_1(G_i), \tag{4.5a}$$

$$\Omega_i = \{x \in \Omega:\ G_i(x) > 0,\ H_i(x) \geqslant 0\}. \tag{4.5b}$$

Remark 4.5. Suppose that there are only two alleles.

(a) The overdominance condition (4.4) implies the existence of an internal equilibrium. In this case, however, (1.21), (1.3), and (4.4) yield $(i = 1, 2)$

$$H_i(x) = r_{ii}(x) - r_{12}(x) < 0, \tag{4.6}$$

so $\Omega_i = \emptyset$, as expected for overdominance.

(b) It is instructive to display the sets Ω_1 and Ω_2. From (1.21) and (4.3) we find

$$\Omega_1 = \{x \in \Omega:\ r_{11}(x) \geqslant r_{12}(x) > r_{22}(x)\}, \tag{4.7a}$$

$$\Omega_2 = \{x \in \Omega:\ r_{22}(x) \geqslant r_{12}(x) > r_{11}(x)\}. \tag{4.7b}$$

See [14] for an asymptotic study of the transition layer of the gene frequency between Ω_1 and Ω_2.

4.2. No dominance

All our theorems simplify greatly under assumption (1.11), i.e., when both frequency dependence and dominance are absent.

Now (A1) reduces to the assumption that

(A1**) there exist $i\in N$ and constants γ_{ij} that satisfy (1.14a) and

$$\sum_j \gamma_{ij}s_j(x)\geqslant s_i(x) \tag{4.8}$$

for every $x\in\bar{\Omega}$, and that the inequality (4.8) is strict for some $x\in\bar{\Omega}$.

Consequently, we have

Corollary 4.6. *If* (1.11) *and* (A1**) *hold, then the conclusions of Theorem* 1.1 *apply.*

We define

$$s_i^*(x)=\max_{j\neq i} s_j(x). \tag{4.9}$$

Since (1.11) implies (A3), Theorem 1.3 yields

Corollary 4.7. *If* (i) (1.11) *holds*, (ii) *there exists* $i\in N$ *such that* $s_i(x)<s_i^*(x)$ *for every* $x\in\bar{\Omega}$, *and* (iii) λ *is sufficiently large, then* $p_i(x,t)\to 0$ *uniformly in* x *as* $t\to\infty$.

Example 4.8 establishes that if the inequality in (ii) is not strict, Corollary 4.7 fails.

Example 4.8. Take $i=n=3$ in Corollary 4.7. Suppose that (1.11) holds and $s_1(x)-s_2(x)$ changes sign (otherwise s_1 and s_2 are arbitrary). Define

$$\Omega_+=\{x\in\bar{\Omega}: s_1(x)\geqslant s_2(x)\},\quad \Omega_-=\{x\in\bar{\Omega}: s_1(x)\leqslant s_2(x)\}. \tag{4.10}$$

For $x\in\Omega_+$, set $s_3(x)=s_1(x)$; for $x\in\Omega_-$, choose $s_3(x)$ such that $s_1(x)\leqslant s_3(x)\leqslant s_2(x)$ and posit that $s_1(x)\not\equiv s_3(x)\not\equiv s_2(x)$. Note that $s_3(x)\leqslant\max\{s_1(x),s_2(x)\}$ in $\bar{\Omega}$, and $s_3(x)\not\equiv\max\{s_1(x),s_2(x)\}$. In view of Corollary 4.7, one might expect that for sufficiently large λ, $p_3(x,t)\to 0$ as $t\to\infty$, since A_3 is nowhere the fittest. A little surprisingly, we have

Claim. *If* λ *is sufficiently large,* $(p_1,p_2,p_3)\to(0,\theta,1-\theta)$ *as* $t\to\infty$, *where* $\theta(x)$ *is the unique solution of the problem*

$$L\theta+\lambda(s_2-s_3)\theta(1-\theta)=0\quad\text{in }\Omega,\quad B\theta|_{\partial\Omega}=0,\ \ 0<\theta<1\quad\text{in }\Omega. \tag{4.11}$$

To establish this assertion, first observe that $s_3\geqslant s_1$ and $s_3\not\equiv s_1$ in $\bar{\Omega}$. By Corollary 4.6, $p_1(x,t)\to 0$ uniformly as $t\to\infty$. The rest of the proof is almost identical to that of Theorem 3.2 of [17], and therefore we omit it.

Turning to Theorem 1.4, from (1.11) and (1.21) we deduce

$$G_i(x) = H_i(x) = s_i(x) - s_i^*(x). \tag{4.12}$$

Therefore, Remark 1.6 and Proposition 4.1 give

Corollary 4.9. *If* (1.11) *holds and* $s_i(x^{(i)}) > s_i^*(x^{(i)})$ *for some* $i \in N$ *and some* $x^{(i)} \in \bar{\Omega}$, *then for every* $\lambda > \lambda_1(G_i)$, *the conclusion of Theorem* 1.4 *applies.*

This means that the allele A_i is protected (a) for every $\lambda > 0$ if $\beta_i \geqslant 0$ and (b) for every $\lambda > \lambda_1(G_i)$ if $\beta_i < 0$. Generically, case (a) can apply to at most one allele. Indeed, for every $k, l \in N$, from (4.12) and (4.9) we get

$$G_k(x) + G_l(x) = [s_k(x) - s_l^*(x)] + [s_l(x) - s_k^*(x)] \leqslant 0,$$

whence (1.22) yields $\beta_k + \beta_l \leqslant 0$. Therefore, there exists at most one $i \in N$ such that $\beta_i > 0$.

Together, Corollaries 4.7 and 4.9 demonstrate that, for sufficiently large λ, unless $s_i(x) \leqslant s_i^*(x)$ for every $x \in \bar{\Omega}$ and $s_i(x^*) = s_i^*(x^*)$ for some $x^* \in \bar{\Omega}$ (cf. Example 4.8), the allele A_i is either globally eliminated or globally protected.

From (4.12) and Remark 4.4 we deduce immediately

Corollary 4.10. *If* (1.11) *holds, then* F_i *can be omitted from Theorems* 1.8 *and* 1.9, *which apply with*

$$\lambda_0 = \max_{i \in N} \lambda_1(G_i), \tag{4.13a}$$

$$\Omega_i = \{x \in \Omega\colon\ s_i(x) > s_i^*(x)\}. \tag{4.13b}$$

Thus, an internal equilibrium $\hat{p}(x)$ exists for sufficiently weak migration if $\Omega_i \neq \emptyset$ for every $i \in N$, and $\hat{p}(x)$ has a step-function limiting profile.

Remark 4.11. It is interesting to compare Corollary 4.10 with a result for a discrete-space, discrete-time, frequency-independent migration-selection model. By Example 2.2 in [21], if there is no dominance and the number of alleles does not exceed the number of colonies, then for sufficiently weak migration, there exists exactly one asymptotically stable internal equilibrium, and it has the expected step-function limiting profile. Similarly, we suspect that in Corollary 4.10, for sufficiently large λ, there exists exactly one internal equilibrium and it is globally attracting.

4.3. Multiplicative selection coefficients

Instead of (1.11), here we suppose

$$r_{ij}(x) = s_i(x)s_j(x) \tag{4.14}$$

for every $i,j\in N$ and every $x\in\bar{\Omega}$. Since translating r_{ij} so that $r_{ij}(x)>0$ for every i,j, and x does not change (1.8), without loss of generality we assume that $s_i(x)>0$ for every i and x. Then (1.12) implies that $\bar{s}(x,p)>0$ for every $x\in\bar{\Omega}$ and every $p\in\Delta$.

From (1.3) we get

$$r_i(x,p)=\bar{s}(x,p)s_i(x), \tag{4.15}$$

so assumption (A1) again reduces to (A1**). In (1.18),

$$r^*(x,p)=\bar{s}(x,p)\max_{j\in N}s_j(x), \tag{4.16}$$

whence (4.15) and (4.9) show that $r_i(x,p)<r^*(x,p)$ if and only if $s_i(x)<s_i^*(x)$. We note also that (4.14) implies (A3). Finally, from (1.21) and (4.14) we obtain

$$G_i(x)=\begin{cases}s_i^{**}(x)[s_i(x)-s_i^*(x)] & \text{if } s_i(x)\geqslant s_i^*(x),\\ s_i^*(x)[s_i(x)-s_i^*(x)] & \text{if } s_i(x)\leqslant s_i^*(x),\end{cases} \tag{4.17a}$$

$$H_i(x)=s_i(x)[s_i(x)-s_i^*(x)], \tag{4.17b}$$

where

$$s_i^{**}(x)=\min_{j\neq i}s_j(x). \tag{4.18}$$

The observations in this paragraph yield

Corollary 4.12. *Corollaries* 4.6, 4.7, 4.9, *and* 4.10 *hold with* (1.11) *and* (4.12) *replaced by* (4.14) *and* (4.17), *respectively.*

4.4. Genotype-independent spatial dependence

In many papers in the literature, frequency independence is specialized by the natural assumption that

$$r_{ij}(x)=g(x)\sigma_{ij}, \tag{4.19}$$

for every i,j, and x, where $g(x)$ changes sign in Ω and the σ_{ij} $(=\sigma_{ji})$ are constants. This means that the direction and intensity of selection depend on position, but the selection pattern (e.g., dominance relations) does not.

Substituting (4.19) into (1.14b) and appealing to the assumption that $g(x)$ changes sign, we infer that

$$\sum_j\gamma_{ij}\sigma_j(p)=\sigma_i(p) \tag{4.20}$$

for every $p\in\Delta$, where

$$\sigma_i(p)=\sum_j \sigma_{ij}p_j. \tag{4.21}$$

Therefore, inequality (1.14b) cannot be strict and (A1) fails. We replace (A1) by the following two assumptions.

(A1.1) There exist $i\in N$ and constants γ_{ij} that satisfy (1.14a) such that (4.20) holds for every $p\in\Delta$.
(A1.2) For every $p\in\Delta$, there exists $k=k(i,p)$ such that $\gamma_{ik}>0$ and $\sigma_k(p)\neq\sigma_i(p)$.

At the end of this subsection, we shall prove

Proposition 4.13. *If* (A1.1) *and* (A1.2) *hold, then the conclusions of Theorem* 1.1 *apply*.

It is easy to see that (4.19) implies the inconsistency of (A2) and (A3) and therefore the failure of Theorem 1.3.

Defining

$$\hat{\sigma}_i=\min_{k\neq i}\sigma_{ik},\quad \check{\sigma}_i=\max_{k\neq i}\sigma_{ik}, \tag{4.22a}$$

$$\sigma_i^{*}=\max_{j\neq i,k\neq i}\sigma_{jk},\quad \sigma_i^{**}=\min_{j\neq i,k\neq i}\sigma_{jk}, \tag{4.22b}$$

from (1.21) and (4.19) we obtain

$$G_i(x)=\begin{cases} g(x)(\hat{\sigma}_i-\sigma_i^{*}) & \text{if } g(x)\geqslant 0,\\ g(x)(\check{\sigma}_i-\sigma_i^{**}) & \text{if } g(x)<0,\end{cases} \tag{4.23a}$$

$$H_i(x)=\begin{cases} g(x)(\sigma_{ii}-\check{\sigma}_i) & \text{if } g(x)\geqslant 0,\\ g(x)(\sigma_{ii}-\hat{\sigma}_i) & \text{if } g(x)<0.\end{cases} \tag{4.23b}$$

We now have

Corollary 4.14. *Remarks* 1.6 *and* 4.4 *apply with the simplification* (4.23).

Remark 4.15. If, in addition to (4.19), we posit the absence of dominance, we have $\sigma_{ij}=\tau_i+\tau_j$ for some constants τ_i, and we label the alleles so that $\tau_1>\tau_2>\cdots>\tau_n$. Now (4.20) reduces to

$$\sum_j \gamma_{ij}\tau_j=\tau_i, \tag{4.24}$$

which holds for every $i\in\tilde{N}$ if we choose $\gamma_{ij}=0$ unless $j=1$ or $j=n$, and

$$\gamma_{i1}=\frac{\tau_i-\tau_n}{\tau_1-\tau_n},\quad \gamma_{in}=1-\gamma_{i1}. \tag{4.25}$$

Furthermore, assumption (A1.2) holds with $k=1$ (and $k=n$) for every $i\in\tilde{N}$. Consequently, Theorem 1.1 applies. In fact, as shown on p. 394 of [17], Theorems 3.2 and 3.3 of [17] fully determine the asymptotic behavior of $p(x,t)$.

We conclude this section with the

Proof of Proposition 4.13. By Lemma 2.1 and (A1.1), $\min_{x\in\bar{\Omega}}u_i(x,t)$ is non-decreasing in t. Hence, in view of the proof of Theorem 1.1, it suffices to show that if (A1.1) and (A1.2) hold, $\min_{x\in\bar{\Omega}}u_i(x,t)$ is a non-constant function of t. We argue by contradiction: suppose that $\min_{x\in\bar{\Omega}}u_i(x,t)$ is equal to some constant, say $\bar{u}_i$, for every $t>0$. Then, as in the proof of Lemma 2.2, we have $u(x,t)=\bar{u}_i$ for every $x\in\bar{\Omega}$ and $t>0$. Furthermore, from (2.2) we get

$$\sum_{1\leqslant k<l\leqslant n}\gamma_{ik}\gamma_{il}\left(\nabla\ln\frac{p_k}{p_l}\right)\cdot V\nabla\ln\frac{p_k}{p_l}\equiv 0 \tag{4.26}$$

for every $x\in\bar{\Omega}$ and every $t>0$.

We claim that if $\gamma_{ik}>0$, then $\sigma_k(p(x,t))=\sigma_i(p(x,t))$ for every $x\in\bar{\Omega}$ and every $t>0$. However, this assertion contradicts (A1.2).

To establish our assertion, we first show that if $\gamma_{ik}>0$ and $\gamma_{il}>0$, then $\sigma_k(p(x,t))=\sigma_l(p(x,t))$ for every $x\in\bar{\Omega}$ and every $t>0$. Define

$$N_i=\{k\in N:\gamma_{ik}>0\}. \tag{4.27}$$

By (1.14a), N_i is non-empty. We observe that N_i contains at least two elements: for otherwise, (1.14a) shows that $\gamma_{ik^*}=1$ for some k^*; this and (4.20) imply that $\sigma_{k^*}(p)=\sigma_i(p)$ for every $p\in\Delta$, which contradicts (A1.2). From (4.26) we see that if $\gamma_{ik}>0$ and $\gamma_{il}>0$, then

$$\left(\nabla\ln\frac{p_k}{p_l}\right)\cdot V\nabla\ln\frac{p_k}{p_l}=0 \tag{4.28}$$

for every $x\in\bar{\Omega}$ and $t>0$. Hence,

$$\frac{p_k}{p_l}=\chi(t) \tag{4.29}$$

for some positive function $\chi(t)$. As on p. 407 of [17], from (4.29) and the equations of p_k and p_l in (1.8a), we obtain

$$\frac{\chi'(t)}{\chi(t)}=\lambda g(x)[\sigma_k(p)-\sigma_l(p)] \tag{4.30}$$

for every $x\in\bar{\Omega}$ and $t>0$. Since $g(x)=0$ for some $x\in\Omega$, therefore $\chi'(t)=0$ for every $t>0$. Since $g(x)\not\equiv 0$, we conclude from (4.30) that

$$\sigma_k(p(x,t))=\sigma_l(p(x,t)) \tag{4.31}$$

for every $x\in\bar{\Omega}$ and every $t>0$. Therefore, (4.31) holds for every $k,l\in N_i$. Then (4.31), (4.20), and (1.14a) imply that for every $k\in N_i$,

$$\sigma_i(p)=\sum_j \gamma_{ij}\sigma_k(p)=\sigma_k(p). \tag{4.32}$$

This establishes our assertion, which completes the proof of Proposition 4.12. □

5. Discussion

Here, we complement our analysis of weak migration with a discussion of strong and intermediate migration.

5.1. Strong migration

For sufficiently small λ, we expect that migration averages the selection coefficients and gene frequencies so that the kinetic (or pure-selection) system approximates (1.8). To support this conjecture, we sketch a formal argument that takes advantage of the separation of time scales: migration and selection act at rates of order 1 and λ, respectively.

Setting $S_i=\lambda\tilde{S}_i$, we rewrite (1.8a) in the vector form

$$p_t=Lp+\lambda\tilde{S}. \tag{5.1}$$

We use an asterisk to signify averages with respect to ψ, defined in Section 1:

$$p^*(t)=(\psi,p(x,t))=\int_\Omega \psi(x)p(x,t)\,dx. \tag{5.2}$$

We have $(\psi,L_1p)=(p,L_1^\dagger\psi)=0$, whence (5.1) yields

$$p_t^*=\lambda(\psi,\tilde{S}). \tag{5.3}$$

Thus, p^* is our slow dependent variable.

Defining $\tau=\lambda t$ and letting $\lambda\to 0$ with $\tau>0$ and fixed in (5.1) suggests that $p(x,t)\to P(\tau)$, independent of x. We now let $\lambda\to 0$ in (5.3) and recall (1.3) and (1.4) to derive

$$\frac{dP_i}{d\tau}=P_i(r_i^*-\bar{r}^*),\quad \tau>0, \tag{5.4a}$$

$$P_i(0)=p_i^*(0), \tag{5.4b}$$

where

$$r_{ij}^* = (\psi, r_{ij}(x,P)), \quad r_i^* = \sum_j r_{ij}^* P_j, \quad \bar{r}^* = \sum_{i,j} r_{ij}^* P_i P_j. \tag{5.4c}$$

At least for $\tau \leqslant T < \infty$, we expect (5.4) to approximate (1.8). If (5.4) has a global attractor, we conjecture that for sufficiently small λ, so does (1.8), and it is approximated by that of (5.4).

A theorem in this direction is that of Carvalho and Hale [5]: if L is in divergence form, then as $\lambda \to 0$, every attractor of (1.8) converges to an attractor of (5.4). By Theorem 4.2 of [17], the system (1.8) can be expressed in divergence form if and only if the quite special equation

$$M\rho^2 - \tfrac{1}{2}\nabla \cdot (V\rho^2) = 0 \tag{5.5}$$

holds in $\bar{\Omega}$. Theorem 4.4 of [17] shows that divergence form applies if the population density $\rho(x)$ is constant and (i) $M = 0$ and $V(x)$ is constant, (ii) migration is conservative (i.e., it does not change the population density), or (iii) migration is symmetric (i.e., the underlying discrete migration pattern is described by a symmetric forward migration matrix). Thus, divergence form is biologically rather restrictive. According to Theorem 4.4 of [17], variational form applies in at least five biologically natural cases. Therefore, extending the result of Carvalho and Hale [5] even to variational form would be biologically important.

5.2. *Intermediate migration*

This is the richest and most difficult regime because if λ is of order unity, then migration and selection are of comparable importance. We suppose that both frequency dependence and dominance are absent, i.e., (1.11) holds.

For two alleles, by Theorem 2.1 of [17], if an internal equilibrium exists, it is globally attracting. For three alleles, by Theorems 3.2 and 3.3 of [17], assumptions (a2) and (a3) imply that if an edge equilibrium exists, it is globally attracting. Work in progress demonstrates that, even with homogeneous, isotropic migration ($L = \nabla^2$), if (a3) is violated, more complicated dynamics can occur: In one example, as λ increases to some critical value, a stable edge solution becomes unstable and a stable or unstable interior equilibrium bifurcates from the edge solution. As λ increases further, this interior equilibrium reaches $\partial\Delta$ and connects with another edge solution, which becomes the global attractor of (1.8) for the next interval of λ. This phenomenon can repeat as λ increases. In fact, the two edge solutions can change stability an arbitrary but finite number of times, and correspondingly, interior equilibria can also exist for an arbitrary but finite number of intervals of λ. These issues will be discussed in a forthcoming paper.

Appendix

Here, we state and prove two propositions. Proposition A exhibits an interesting connection between assumption (A1) and assumptions such as (1.10) that involve a single constant γ_i for each allele. When elimination of all but two alleles is studied, this connection may facilitate the analysis by suggesting the application of simpler sufficient conditions.

Proposition A. *Suppose that assumption* (A1) *holds for every* $i\in\tilde{N}$. *Then for every* $i\in\tilde{N}$, *there exists a constant* $\gamma_i\in[0,1]$ *such that*

$$\gamma_i r_1(x,p)+(1-\gamma_i)r_n(x,p)\geqslant r_i(x,p) \tag{A.1}$$

for every $x\in\bar{\Omega}$ *and every* $p\in\Delta$, *and that the inequality* (A.1) *is strict for some* $x\in\bar{\Omega}$ *and every* $p\in\operatorname{int}\Delta$.

Proposition B gives insight into the assumptions required in Theorem 1.4 for protection of an allele. The condition that $F_i(x)>0$ for some $x\in\bar{\Omega}$ is equivalent to the assumption that

(A4) there exist $i\in N$ and $x^{(i)}\in\bar{\Omega}$ such that $r_i(x^{(i)},p)>\max_{j\neq i}r_j(x^{(i)},p)$ for every $p\in\Delta$.

Similarly, the condition that $G_i(x)>0$ for some $x\in\bar{\Omega}$ is equivalent to the assumption that

(A4*) there exist $i\in N$ and $x^{(i)}\in\bar{\Omega}$ such that $r_{ik}(x^{(i)},p)>\max_{j\neq i}r_{jk}(x^{(i)},p)$ for every $k\neq i$ and every $p\in\Delta$.

Now consider the restriction of assumption (A4) to the face of $\Delta^{(i)}$ of Δ:

(A4**) There exist $i\in N$ and $x^{(i)}\in\bar{\Omega}$ such that $r_i(x^{(i)},p)>\max\limits_{j\neq i}r_j(x^{(i)},p)$ for every $p\in\Delta^{(i)}$.

Proposition B. *If* (1.9) *holds*, *assumptions* (A4*) *and* (A4**) *are equivalent.*

A.1. Proof of Proposition A

We use induction on n. Our assertion obviously holds for $n=3$. Suppose that it holds for n. We prove that it holds also for $n+1$, i.e., assuming that there exist constants γ_{ij} $(2\leqslant i\leqslant n,\ 1\leqslant j\leqslant n+1)$ such that

$$\gamma_{ij}\geqslant 0,\quad \gamma_{ii}=0,\quad \sum_{j=1}^{n+1}\gamma_{ij}=1, \tag{A.2a}$$

$$\sum_{j=1}^{n+1} \gamma_{ij} r_j(x,p) \geqslant r_i(x,p) \tag{A.2b}$$

for every $x \in \bar{\Omega}$ and every $p \in \Delta$, and that there exists some $x \in \bar{\Omega}$ such that inequality (A.2b) is strict for every $p \in \operatorname{int} \Delta$, Proposition A holds with n replaced by $n+1$.

We first claim that, for every $k, l \in \{2, 3, \ldots, n\}$,

$$\gamma_{kl}\gamma_{lk} < 1. \tag{A.3}$$

To show that (A.3) holds, we argue by contradiction: suppose that there exist $k^*, l^* \in \{2, \ldots, n\}$ such that $\gamma_{k^*l^*}\gamma_{l^*k^*} = 1$, i.e., $\gamma_{k^*l^*} = \gamma_{l^*k^*} = 1$. Since $\gamma_{ii} = 0$ for every i, we see that $k^* \neq l^*$. Thus, $\gamma_{k^*l^*} = 1$ implies that $\gamma_{k^*j} = 0$ for every $j \neq l^*$. Setting $i = k^*$ in (A.2b), we get

$$r_{l^*}(x,p) \geqslant r_{k^*}(x,p) \tag{A.4}$$

for every $x \in \bar{\Omega}$ and every $p \in \Delta$, and (A.4) is strict for some x and every $p \in \operatorname{int} \Delta$.

Similarly, $\gamma_{l^*k^*} = 1$ implies that $\gamma_{l^*j} = 0$ for every $j \neq k^*$. Setting $i = l^*$ in (A.2b), we get

$$r_{k^*}(x,p) \geqslant r_{l^*}(x,p) \tag{A.5}$$

for every $x \in \bar{\Omega}$ and every $p \in \Delta$. Hence, $r_{k^*}(x,p) \equiv r_{l^*}(x,p)$ for every $x \in \bar{\Omega}$ and $p \in \Delta$. However, this contradicts that (A.4) is strict for some x. This proves (A.3).

Letting $l = n$ in (A.3), we get

$$\gamma_{kn}\gamma_{nk} < 1 \tag{A.6}$$

for every $k \in \{2, \ldots, n\}$. Putting $i = n$ in (A.2b), we obtain

$$r_n(x,p) \leqslant \sum_{j=1}^{n+1} \gamma_{nj} r_j(x,p) = \sum_{1 \leqslant j \leqslant n+1,\, j \neq n} \gamma_{nj} r_j(x,p). \tag{A.7}$$

For every $i \in \tilde{N}$, from (A.2b) and (A.7) we have

$$\begin{aligned} r_i(x,p) &\leqslant \sum_{1 \leqslant j \leqslant n+1,\, j \neq n} \gamma_{ij} r_j(x,p) + \gamma_{in} r_n(x,p) \\ &\leqslant \sum_{1 \leqslant j \leqslant n+1,\, j \neq n} (\gamma_{ij} + \gamma_{in}\gamma_{nj}) r_j(x,p) \\ &= \sum_{1 \leqslant j \leqslant n+1,\, j \neq i,n} (\gamma_{ij} + \gamma_{in}\gamma_{nj}) r_j(x,p) + \gamma_{in}\gamma_{ni} r_i(x,p), \end{aligned} \tag{A.8}$$

which implies that, for every $i \in \tilde{N}$,

$$(1 - \gamma_{in}\gamma_{ni}) r_i(x,p) \leqslant \sum_{1 \leqslant j \leqslant n+1,\, j \neq i,n} (\gamma_{ij} + \gamma_{in}\gamma_{nj}) r_j(x,p). \tag{A.9}$$

From (A.6) and (A.9) we see that, for every $i\in\tilde{N}$,

$$r_i(x,p)\leqslant\sum_{j=1}^{n+1}\tilde{\gamma}_{ij}r_j(x,p) \tag{A.10}$$

for every $x\in\bar{\Omega}$ and every $p\in\Delta$, where $\tilde{\gamma}_{ii}=\tilde{\gamma}_{in}=0$ for every i, and

$$\tilde{\gamma}_{ij}=\frac{\gamma_{ij}+\gamma_{in}\gamma_{nj}}{1-\gamma_{in}\gamma_{ni}} \tag{A.11}$$

for every $j\in\{1,\ldots,n+1\}$ such that $j\neq i,n$. From (A.11) and (A.2a) it follows that, for every $i\in\tilde{N}$,

$$\sum_{j=1}^{n+1}\tilde{\gamma}_{ij}=1, \tag{A.12}$$

and $\tilde{\gamma}_{ij}\geqslant 0$ for every $j\in\{1,\ldots,n+1\}$ such that $j\neq n$. Furthermore, the above argument demonstrates that, for every $i\in\tilde{N}$, (A.10) holds with $\tilde{\gamma}_{in}=0$ and is strict for some $x\in\bar{\Omega}$ and every $p\in\operatorname{int}\Delta$. Iterating this argument leads to (A.1) with n replaced by $n+1$, and induction on n completes the proof of Proposition A. □

A.2. Proof of Proposition B

If (1.9) holds, then by choosing p at each of the $n-1$ vertices of the face $\Delta^{(i)}$, we see that (A4**) reduces to (A4*).

If (1.9) and (A4*) hold, we have $r_{ik}(x^{(i)})>r_{jk}(x^{(i)})$ for every $j\neq i$ and $k\neq i$. Therefore, for every $j\neq i$ and every $p\in\Delta^{(i)}$,

$$\begin{aligned} r_i(x^{(i)},p)-r_j(x^{(i)},p)&=\sum_{1\leqslant k\leqslant n,\,k\neq i}[r_{ik}(x^{(i)})-r_{jk}(x^{(i)})]p_k\\ &\geqslant\left\{\min_{1\leqslant k\leqslant n,\,k\neq i}[r_{ik}(x^{(i)})-r_{jk}(x^{(i)})]\right\}\sum_{1\leqslant k\leqslant n,\,k\neq i}p_k\\ &=\min_{1\leqslant k\leqslant n,\,k\neq i}[r_{ik}(x^{(i)})-r_{jk}(x^{(i)})]>0. \end{aligned} \tag{A.13}$$

Hence, (A4**) holds. □

References

[1] R.S. Cantrell, C. Cosner, Spatial Ecology via Reaction–Diffusion Equations, in: Series in Mathematical and Computational Biology, Wiley, Chichester, UK, 2003.

[2] R.S. Cantrell, C. Cosner, V. Hutson, Permanence in ecological systems with diffusion, Proc. Roy. Soc. Edinburgh A123 (1993a) 553–559.

[3] R.S. Cantrell, C. Cosner, V. Hutson, Permanence in some diffusive Lotka–Volterra models for three interacting species, Dyn. System Appl. 2 (1993b) 505–530.
[4] R.S. Cantrell, C. Cosner, V. Hutson, Ecological models, permanence and spatial heterogeneity, Rocky Mount. J. Math. 26 (1996) 1–35.
[5] A.N. Carvalho, J.K. Hale, Large diffusion with dispersion, Nonl. Anal. TMA 17 (1991) 1139–1151.
[6] S.R. Dunbar, K.P. Rybakowski, K. Schmitt, Persistence in models of predator–prey populations with diffusion, J. Diff. Eqs. 65 (1986) 117–138.
[7] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equation of Second Order, 2nd Edition, Springer, Berlin, 1983.
[8] J.K. Hale, P.E. Waltman, Persistence in infinite dimensional systems, SIAM J. Appl. Math. 20 (1989) 388–395.
[9] D. Henry, Geometric Theory of Semilinear Parabolic Equations, in: Lecture Notes in Mathematics, Vol. 840, Springer, Berlin, 1981.
[10] V. Hutson, A theorem on average Liapunov functions, Monatsh. Math. 98 (1984) 267–275.
[11] V. Hutson, The existence of an equilibrium for permanent systems, Rocky Mount. J. Math. 20 (1990) 1033–1040.
[12] V. Hutson, W. Moran, Repellers in reaction–diffusion systems, Rocky Mount. J. Math. 17 (1987) 301–314.
[13] V. Hutson, K. Schmitt, Permanence in dynamical systems, Math. Biosci. 8 (1992) 1–71.
[14] J.B. Keller, Genetic variability due to geographical inhomogeneity, J. Math. Biol. 20 (1984) 223–230.
[15] M.G. Krein, M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Transl. 26 (1950) 1–128.
[16] C.S. Lin, W.M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72 (1988) 1–27.
[17] Y. Lou, T. Nagylaki, A semilinear parabolic system for migration and selection in population genetics, J. Differential Equations 181 (2002) 388–418.
[18] T. Nagylaki, Conditions for the existence of clines, Genetics 80 (1975) 595–615.
[19] T. Nagylaki, The diffusion model for migration and selection, in: A. Hastings (Ed.), Some Mathematical Questions in Biology, Lectures on Mathematics in the Life Sciences, Vol. 20, American Mathematical Society, Providence, RI, 1989, pp. 55–75.
[20] T. Nagylaki, The diffusion model for migration and selection in a dioecious population, J. Math. Biol. 34 (1996) 334–360.
[21] T. Nagylaki, Y. Lou, Patterns of polymorphism maintained by migration and selection, Theor. Popul. Biol. 59 (2001) 297–313.
[22] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum, New York, 1992.
[23] M.H. Protter, H.F. Weinberger, Maximum Principles in Differential Equations, 2nd Edition, Springer, Berlin, 1984.
[24] R. Redlinger, Über die C^2-Kompaktheit der Bahn von Lösungen semilinearer parabolischer Systeme, Proc. Roy. Soc. Edinburgh A93 (1983) 99–103.
[25] S. Senn, On a nonlinear elliptic eigenvalue problem with Neumann boundary conditions, with an application to population genetics, Comm. Partial Differential Equations 8 (1983) 1199–1228.
[26] S. Senn, P. Hess, On positive solutions of a linear elliptic eigenvalue problem with Neumann boundary conditions, Math. Ann. 258 (1982) 459–470.